and

$$\xi = \eta/C^{1/2} \tag{22}$$

where C is positive and real and $C^{1/2} > 0$.

It is now possible to derive a skin-friction coefficient $C_f(N)$ as defined by Ref. 5 for Newtonian flow

$$C_f(N) = \tau_0{}^* Re_t{}^{1/(N+1)} = \frac{[-F'(0)]^N}{[2N(N+1)]^{N/(N+1)}} \tag{23}$$

in which $\tau_0{}^* \equiv \tau_0/\rho U_0{}^2$ is the dimensionless shear stress at the wall. Re_t is the Reynolds number in terms of time which can be expressed in a form equivalent to the conventional expression for non-Newtonian flow if we replace t by L/U_0 where L is some characteristic length, such that

$$Re_t = \rho U_0{}^2 t^N/K = \rho U_0{}^{2-N} L^N/K$$

Note that the velocity gradient at the wall $F'(0)$ in Eq. (23) is always negative and can be obtained from Eq. (10) such that

$$[-F'(0)] = [(1-N)C]^{1/(N-1)}, \quad \text{where} \quad C = C(N) \tag{24}$$

The dimensionless boundary-layer thickness defined at $u/U_0 = 0.01$ can now be obtained from Eqs. (3a) and (6) as

$$\delta^* \equiv (y/U_0 t) Re_t{}^{1/(N+1)} = [2N(N+1)]^{1/(N+1)} \eta|_{F(\eta)=0.01} \tag{25}$$

Impulsively Started Flow (ISF)

With a technique similar to that developed for the ISP case, an impulsively started power law fluid flow with a constant velocity U_0 over a stationary plate has the following velocity distribution function

$$f'(\eta) = \int_0^\xi (1-\xi^2)^{1/(N-1)} d\xi \Big/ \int_0^1 (1-\xi^2)^{1/(N-1)} d\xi = 1 - F(\eta) \quad \text{for} \quad N > 1 \tag{26}$$

and

$$f'(\eta) = \int_0^\xi \frac{1}{(1+\xi^2)^{1/(1-N)}} d\xi \Big/ \int_0^\infty \frac{1}{(1+\xi^2)^{1/(1-N)}} d\xi = 1 - F(\eta) \quad \text{for} \quad N < 1 \tag{27}$$

where $f'(\eta) = u/U_0$, $\xi = \eta/|C|^{1/2}$, and $|C|$ is the absolute value of C which can be obtained from Eqs. (15) and (21) for $N \gtrless 1$. The skin-friction coefficient can be obtained by replacing $-F'(0)$ with $f''(0)$ in Eq. (23) and the dimensionless boundary-layer thickness is defined at $u/U_0 = 0.99$ for η in Eq. (25).

Discussion

The dimensionless velocity distributions for various power law fluids are calculated and plotted in Fig. 1. The velocity increases for ISF or decreases for ISP and decays much faster when the power law fluid index is higher. The important flow parameters such as skin-friction coefficient C_f, velocity gradient at the wall $f''(0)$ for ISF and $F'(0)$ for ISP, dimensionless boundary-layer thickness δ^*, finite boundary-layer thickness η_∞ and constants B and C are tabulated in Table 1 for various values of N. Figure 1 and Table 1 utilize the similarity transformation of Eq. (6). The results for pseudoplastic fluids are in agreement with those of Ref. 1. A solution of Eq. (5), which utilizes the similarity transformation of Eq. (4), has been obtained and is plotted for reference in Fig. 2.

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Optimization of Simple Structures with Higher Mode Frequency Constraints

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Nomenclature

$t(x)$ = $T(x)/T_{ref}$, nondimensional thickness
$t_m(x)$ = nondimensional thickness associated with the mth frequency
x = nondimensional coordinate
α_m = $\delta[(2m-1)/2]^2\pi^2$
β_m = $(1-\delta_1)[(2m-1)/2]^2\pi^2$
δ_1 = ratio between reference structural weight and total reference structure weight, a constant
$\theta(x)$ = cylinder cross section angle of rotation
$(\ )'$ = $d(\ )/dx$

Introduction

THIS Note presents some interesting results of a study of least weight optimization of simple structures with a single natural frequency constraint. This frequency constraint may be any one of the natural frequencies of the structure. Variational techniques are used to derive the necessary equations. Numerical solution methods have been used, where necessary, to find solutions to the nonlinear, two-point boundary value problems which result.

Discussion

Variational calculus provides a convenient and useful tool for the study of simple structural optimization problems. Although this method of analysis is somewhat restricted, the results of such analyses often provide insight into more sophisticated problems. This approach to the search for a least weight structure whose fundamental frequency is fixed at a certain value has been the subject of several articles.[1–4] Various methods have been employed to achieve solutions to this class of problems. In many cases, although the governing equations are easily derived, an analytic solution cannot be found. However, several numerical techniques have been presented which overcome this difficulty. The free torsional vibration of a thin-wall cylinder of length L with variable wall thickness $T(x)$ is discussed in detail by Armand.[5] The necessary conditions for a least weight cylinder with a fixed fundamental torsional frequency are also derived in Ref. 5. A simple extension of Armand's work shows that free vibration in the mth mode can be described by the nondimensional differential equations (1a–c). A nonstructural mass moment of inertia has been added to ensure that the problem is properly posed.

$$\theta_m'(x) = s_m/t \qquad 0 < x < 1 \tag{1a}$$

$$s_m'(x) = -(\alpha_m t + \beta_m)\theta_m(x) \tag{1b}$$

and

$$\theta_m(0) = s_m(1) = 0 \tag{1c}$$

Only the nondimensional wall thickness $t(x)$ is free to be varied. An objective function is defined as

$$J = \int_0^1 t(x)\,dx \tag{2}$$

The problem is to find a function $t(x)$ which yields a stationary value of J and satisfies equations (1a–c). A straightforward extension of this problem is that of fixing several frequencies simultaneously.[6]

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The previously cited references discuss equivalent methods of finding necessary conditions for this problem. The equation for the thickness is shown to be

$$t^2(x) = (s_m)^2/(1+\alpha_m \theta_m{}^2) \tag{3}$$

Equation (3) together with Eqs. (1a–c) gives the necessary conditions for a stationary value of the objective function J. An additional element of realism may be added by specifying an inequality constraint involving the nondimensional thickness

$$t(x) \geqq t_{\min} \tag{4}$$

Equation (4) requires that if $t(x)$, as determined from Eq. (3), is less than $t_{\min}$, then the value $t(x)$ used in Eqs. (1a and b) must be equal to $t_{\min}$. Reference 5 discusses the analytical solution to the preceding problem with m equal to unity and for various values of $t_{\min}$ and δ_1. The analytic solutions for θ_1, s_1, and $t(x)$ involve hyperbolic sines and cosines as well as sines and cosines. These solutions may be used to rigorously establish the validity of the following discussion.

If only the second torsional frequency is held fixed ($m = 2$), the analytic solution is not readily apparent. Discontinuities in $\theta'(x)$ occur if $s(x)$ changes sign and no $t_{\min}$ is provided. A closed form solution to the $m = 2$ problem does, in fact, exist. However, it was determined only after numerical computation suggested its existence. The necessary conditions were programed for the computer and a "transition matrix" solution[7] was used to solve for the optimal thickness distribution. The results of this calculation are shown in Fig. 1. This figure compares the thickness distribution for m equal to 2 with the solution for m equal to unity. The mass ratio MR expresses the ratio between the optimum cylinder weight and the weight of a similar cylinder with uniform wall thickness

$$MR = \delta_1 J + (1-\delta_1) \tag{5}$$

The mass ratio for $m = 2$ was found to be identical to that for $m = 1$. The parameters δ_1 and $t_{\min}$ are the same in each case. In addition, the thickness distribution for $m = 2$ appears to be a periodic extension, on a compressed scale, of the $m = 1$ solution.

These numerical results suggested a similarity between the fixed fundamental frequency solution (called the "fundamental solution") and solutions in which a single frequency other than the fundamental is held fixed. These similarities are shown to occur under special circumstances. The reason for these similarities can be found by studying the equations for $m = 1$. An extended solution to the $m = 1$ equations may be constructed analytically. This solution has the properties that $\theta_1(x)$ is zero for $x = 0, 2, 4, \ldots$ and $s_1(x)$ is zero for $x = 1, 3, 5, \ldots$. If $t_{\min} = 0$, the solution for $\theta_1(x)$ has discontinuities at $x = 1, 2, \ldots$. At these points the function $\theta_1(x)$ is continuous but its derivative is discontinuous. Therefore, instead of the mode shape $\theta_1(x)$ being smooth, it has cusps at these points. If $t_{\min} > 0$, then these cusps disappear and both $\theta_1(x)$ and its first derivative are continuous at $x = 1, 2, \ldots$. Thus the period of variables $\theta_1(x)$ and $s_1(x)$ is 4. The expression for $t(x)$ will have

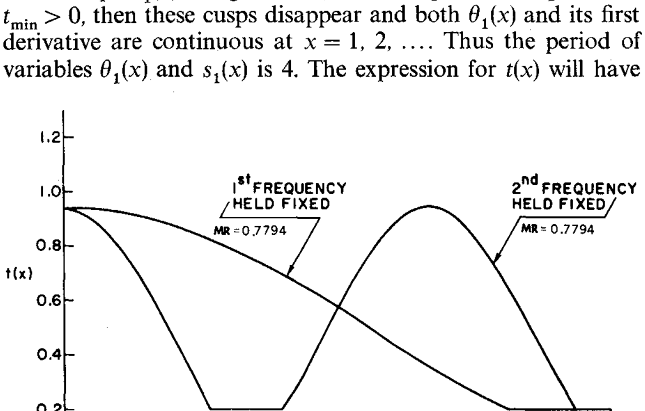

Fig. 1 Nondimensional thickness distributions, minimum weight thin-wall cylinder, first or second frequencies held constant—$\delta_1 = 0.50$, $t_{\min} = 0.20$.

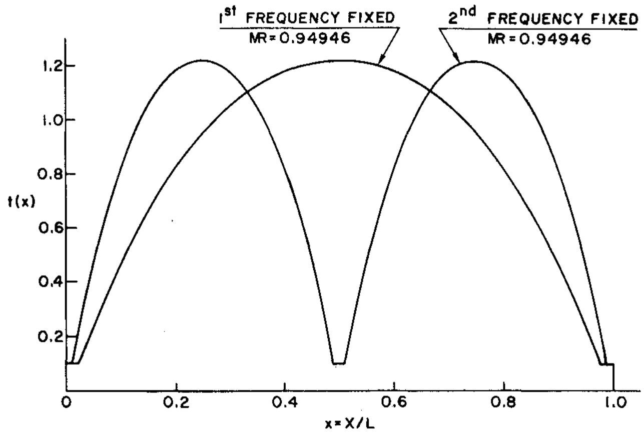

Fig. 2 Nondimensional thickness distributions, minimum weight sandwich beam on simple supports, first or second flexural frequency held constant—$\delta_1 = 0.30$, $t_{\min} = 0.10$.

an identical period. The reader is again referred to Ref. 5 for the general solutions for θ_1, s_1, and t. It may be noted that the mode shape for the uniform thickness cylinder is a sine function of period 4. An examination of Eqs. (1a–c and 3) shows that the solution

$$\theta_m(x) = \theta_1(nx)/n \tag{6a}$$

$$s_m(x) = s_1(nx) \tag{6b}$$

$$t_m(x) = t_1(nx) \qquad (n = 2m-1) \tag{6c}$$

satisfies those equations since $\theta_1(x)$ and $s_1(x)$ are periodic such that

$$s_1(n) = \theta_1(n-1) = 0 \qquad (n = 1, 3, 5 \ldots) \tag{7}$$

This explains why the numerical solution for $m = 2$ behaves as it does.

In this case, the knowledge of the fundamental solution allows one to calculate the solution to problems with constraints on frequencies other than the fundamental. The evaluation of the objective function for these other frequency constraints yields an interesting result.

$$J_m = \int_0^1 t_m(x)\,dx \tag{8a}$$

$$(n)\int_0^1 t_m(x)\,dx = \int_0^1 t_1(nx)\,d(nx) \tag{8b}$$

$$\int_0^1 t_1(nx)\,d(nx) = \int_0^n t_1(x)\,dx \tag{8c}$$

Because of the form of the equation for $t^2(x)$ and since $t(x) \geqq 0$, always, the extended solution for $t(x)$ will have the property that

$$\int_0^1 t_1(x)\,dx = \int_n^{n+1} t_1(x)\,dx \qquad n = 1, 2, \ldots \tag{8d}$$

Therefore,

$$J_m = n\left[\int_0^1 t_1(x)\,dx\right]\Big/ n \tag{8e}$$

and

$$J_m = J_1 \tag{8f}$$

Therefore, the weight savings when the mth natural frequency is fixed is identical to the weight savings when the fundamental is fixed. This statement holds for any value of δ_1 or $t_{\min}$.

A similar situation occurs in some beam optimization problems. The thickness distribution for a minimum weight sandwich beam on simple supports whose fundamental flexural frequency is held fixed is shown in Fig. 2. Also shown is the optimum thickness distribution for a similar problem with only the second bending frequency fixed. The mass ratio in each case is identical and the thickness distributions are similar. In general, for this problem

$$t_m(x) = t_1(mx) \tag{9}$$

In Eq. (9), m refers to the mth natural frequency which is held fixed. Relations similar to those in Eqs. (6a–c) are presented in Ref. 6. This situation occurs because, as in the previous example, the fundamental solution is periodic. The validity of this hypothesis in this problem has been established rigorously for any integer m.

A problem where the previous discussion does not hold true is that of the least weight cantilever beam. The boundary conditions are such that the fundamental solution cannot be made to satisfy boundary conditions for m greater than unity.

Conclusion

It seems reasonable that, if the reference structure in a dynamic optimization problem has eigenfunctions which are periodic, then the least weight structure might also have periodic eigenfunctions. If this is, in fact, the case and if the eigenvalues are also integer multiples of one another, solutions to additional single frequency constraint problems might be generated from "fundamental solutions." If this is true, the value of the objective function which gives an indication of weight savings will not be a function of which natural frequency is held fixed. Although this observation has not been proved rigorously for a general class of problems, it has been illustrated for two simple structures whose natural frequencies are fixed. The subject certainly merits additional investigation.

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Entrainment Equation for Three-Dimensional, Compressible, Turbulent Boundary Layers

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Nomenclature

F = the dimensionless rate of entrainment defined by Eq. (2).
h_1, h_3 = metric coefficients for, respectively, curvilinear coordinates x_1 and x_3.
$\bar{H}$ = $\int_0^\delta \frac{\rho}{\rho_e}\left(1-\frac{u}{u_e}\right)dy\Big/\delta_{11}$
H_{2k} = $\int_0^\delta \frac{u}{u_e}\,dy\Big/\int_0^\delta \frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy$
M = Mach number
s = arc length measured along the x_1 axis according to $ds = h_1\,dx_1$.
u, v, w = boundary-layer velocity components in the x_1, x_2, x_3 directions, respectively.
x_1, x_2, x_3 = orthogonal curvilinear coordinates based on the projections of the outer flow streamlines on the body surface as defined in Fig. 2 of Ref. 5.
y = arc length measured along the x_2 axis according to $dy = dx_2$.
z = arc length measured along the x_3 axis according to $dz = h_3\,dx_3$.
δ = boundary-layer thickness.
δ_1 = $\int_0^\delta \left(1-\frac{\rho u}{\rho_e u_e}\right)dy$
δ_{11} = $\int_0^\delta \left(1-\frac{u}{u_e}\right)\frac{\rho u}{\rho_e u_e}\,dy$
ρ = density
θ_3 = $\int_0^\delta \frac{\rho w}{\rho_e u_e}\,dy$

Subscripts

e = conditions at the outer edge of the boundary layer.
w = conditions at the wall.

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Introduction

THE entrainment theory of Head[1] was originally developed for predicting the flow characteristics of incompressible, turbulent boundary layers. Green[2] extended entrainment theory to compressible, turbulent boundary layers and showed that more accurate results could be obtained by a direct approach as opposed to a transformation to a corresponding incompressible flow. Subsequently, Sumner and Shanebrook[3] modified Green's entrainment theory such that it relies less on empiricism and represents a simpler and more direct extension of Head's incompressible theory to compressible flows. However, Refs. 1–3 are restricted to two-dimensional flow conditions.

Entrainment theory has also been applied to three-dimensional, incompressible, turbulent boundary layers by Cumpsty and Head[4] and by Shanebrook and Hatch[5] who assumed the functions‡ $F(H_{2k})$ and $H_{2k}(\bar{H})$ could be approximated by relations previously developed for two-dimensional flows. The purpose here is to present the entrainment equation for three-dimensional, compressible, turbulent boundary layers and to demonstrate the applicability of relations for $F(H_{2k})$ and $H_{2k}(\bar{H})$, originally developed for two-dimensional incompressible flows, to three-dimensional, compressible, turbulent boundary layers.

Three-Dimensional, Compressible, Entrainment Equation

In an orthogonal curvilinear coordinate system based on the projections of the outer flow streamlines on the body surface, the continuity equation for steady, compressible, turbulent flow may be written in the form[6]

$$[\partial(\rho h_3 u)/\partial x_1] + [\partial(\rho h_1 w)/\partial x_3] + h_1 h_3\,\partial(\rho v)/\partial x_2 = 0 \qquad (1)$$

where the flow variables appear as time-averaged quantities. In the usual fashion Eq. (1) can be integrated across the boundary

‡ The authors have attempted to develop a system of notation that is consistent for both incompressible and compressible, three-dimensional turbulent boundary layers. Thus, the notation adopted here represents a compressible form of the notation used in Ref. 5. The reader should then note that H_{2k} is equivalent to H_2 of Ref. 5 and $\bar{H}$ becomes H_1 of Ref. 5 for incompressible flow conditions.